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A Multiplication Theorem for Room Squares

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A survey of developments in Room Square theory is given, and it is proved that a Room Square of side m can be combined with a Room Square of side n to produce a Room Square of side mn.

1. INTRODUCTION

Room Squares were introduced to mathematicians by an example in [11]. However, it has been pointed out to us by N. S. Mendelsohn and B. Wolk that, under the name of Howell Movements, examples of Room Squares have been familiar in Duplicate Bridge for a long time. The original examples seem to be due to E. C. Howell in 1897 (see [3] and [5]).

Basically, construction of a Room Square involves filling in the cells of a square of side $2n - 1$. In each cell, one either leaves a blank or places an unordered pair of symbols chosen from $2n$ symbols which we usually denote, for convenience, by $\infty, 1, 2, 3,..., 2n - 1$. Each row and each column contains n occupied and $n - 1$ blank cells; also, the n occupied cells in each row and in each column must employ each of the $2n$ possible symbols exactly once. Finally, the $n(2n - 1)$ occupied cells must use each of the $\binom{2n}{2}$ doublets formable from the $2n$ symbols exactly once.

2. SURVEY OF PRESENT KNOWLEDGE

A Room Square of side 1 is trivial, and it is obvious that a square of side 3 does not exist (place $\infty 1$ in any position; then the row containing this doublet must contain 23, and the column containing $\infty 1$ must also contain 23; this is a contradiction). Since no simple proof of the non-existence of a square of side 5 seems to have been published, we include the demonstration we have found most direct.

The Room property is unaltered by permuting columns; so we may assume that the first three cells of the first row of a square of side five are filled, and may name the doublets (see Fig. 1) as ∞, 23, 45. Similarly, we may permute rows so that the first three cells of column one are occupied. The doublets may be taken as 24 and 35 (25 and 34 are equivalent by permuting 4 and 5).

$\infty 1$	23	45		
24				
35				
			25	
				34

FIGURE 1

$\infty 1$	23	45		
24			$\infty 3$	
35				
	$\infty 4$	13	25	
	15	$\infty 2$		34

FIGURE 2

We now ask where the symbols 25 and 34 may appear; since they cannot appear in any of the first three rows and columns, they must appear in the 2×2 square situated in the lower right corner. By permuting rows and columns, we may place 25 in position (4, 4). Then 34 can not appear in positions (4, 5) or (5, 4), since this would require $\infty 1$ to appear in row 4 or column 4; hence 34 appears in position (5, 5), and this forces positions (4, 5) and (5, 4) to be blank.

We next move to Figure 2. Since we now know that all the positions (4, 2), (4, 3), (5, 2), (5, 3), (2, 4), (2, 5), (3, 4), (3, 5) are occupied, it follows that cells (2, 2), (2, 3), (3, 2), (3, 3) are blank. And we must place, in cell (4, 1), a doublet chosen from ∞, 1, 4. Only $\infty 4$ and 14 are possible, and they are equivalent under permutation of the symbols 1 and ∞. So place $\infty 4$ in cell (4, 1), and complete rows 4 and 5.

We now need to place $\infty 3$, and it can only go in cell (2, 4). But $\infty 5$ can only go in cell (2, 5), and this is a contradiction. So a Room Square of side 5 can not exist. The Room Squares of sides 3 and 5 seem to be exceptional, since there is impressive evidence that Room Squares of every odd side n exist ($n \neq 3$, $n \neq 5$).

A limited class of constructions for Room Squares was given in [1] and [2]. The use of Room Squares in statistics is described in [2], and the analysis of such Room Designs, with applications, is given there (but see [12], for a correction to the analysis of variance). The Room Square of side 9 was given in [15], and the first extensive construction of Room Squares was given in [14], where squares of all odd orders 7, 9, 11, 13,..., 47, 49, were exhibited. An individual construction for a square of side 13 was given in [10].

It was stated in [4] that a Room Square of side m and a Room Square of side n could be used to produce a Room Square of side mn. However, the proof employed the concept of Room quasi-groups, and was shown to be invalid in [7]. Recently, as a sequel to [7], Mullin and Nemeth made important constructive advances in [8] and [9]. Basically, they proved that Room Squares exist for all prime-power sides p^k, except possibly when $k = 1$ and p is a Fermat prime $2^{2^t} + 1$ (the Fermat primes are 3, 5, 17, 257, 65537,... ; only five such numbers are known to be prime, and it is known [14] that a square of side 17 exists).

In the light of this theorem by Mullin and Nemeth, it becomes important for the construction of Room Squares to show that they do possess a multiplicative property. In this paper, we consider the problem of constructing a Room Square of side mn, and show that, despite the failure of the quasi-group approach, one still has the valid

MULTIPLICATION THEOREM. *If Room Squares of sides m and n exist, then a Room Square of side mn exists.*

An illustration of this theorem is found in [13]. We might add that, since the current paper was written, a large number of Room Squares have been constructed [6] by extension methods. So the results of the current paper, together with those of [6], leave only a few undecided square sizes below 1000.

3. Proof of the Multiplication Theorem

We combine the construction and the proof.

Let M and N be Room Squares based on the symbols 0, 1, 2,..., m, and 0, 1, 2,..., n, respectively. Let L_1 and L_2 be a pair of orthogonal Latin Squares of order n based on the symbols 1, 2,..., n. Let M^1 be an $m \times m$ square, and fill in the cells of M^1 with squares of side $n \times n$ according to the following prescription:

(a) If cell (i, j) of M is empty, place an $n \times n$ empty square in cell (i, j) of M^1.

(b) If cell (i, j) of M is occupied by the doublet $\{0, k\}$, add kn to each non-zero entry in N, and place the resulting square in cell (i, j) of M^1.

(c) If cell (i, j) of M is occupied by the doublet $\{a, b\}$, where neither a nor b is zero, add an to each entry in L_1, bn to each entry in L_2, superimpose the resulting squares to form an $n \times n$ square completely filled by doublets, and place this square in cell (i, j) of M^1.

Let R denote the $mn \times mn$ square formed by this construction. It is based on the $mn + 1$ symbols $0, n + 1, n + 2, \ldots, n + mn$, and each cell of R contains either two distinct symbols or no symbols.

Since each of the symbols $0, 1, 2, \ldots, m$, occurs exactly once in each row and column of M, the method of constructing R guarantees that each of the symbols $0, n + 1, \ldots, n + mn$, occurs exactly once in each row and in each column of R. Furthermore, if $P(i, j)$ denotes the set of all pairs which occur in the square placed in cell (i, j) of M^1, then all pairs in $P(i, j)$ are distinct from one another. Also, if $(i, j) \neq (s, t)$, it follows that $P(i, j)$ and $P(s, t)$ have no pairs in common. Hence, distinct cells of R are occupied by distinct pairs, and so R is a Room Square. This completes the Multiplication Theorem.

References

1. J. W. Archbold, A combinatorial problem of T. G. Room, *Mathematika* **7** (1960), 50–55.
2. J. W. Archbold and N. L. Johnson, A construction for Room's squares and an application in experimental design, *Ann. Math. Statist.* **29** (1958), 219–225.
3. G. Beynon, "Bridge Director's Manual for Duplicate Games," George Coffin, Waltham, Mass., 1943.
4. R. H. Bruck, What is a loop?, "Studies in Modern Algebra" (ed., A. A. Albert), Prentice-Hall, Englewood Cliffs, N. J., 1963.
5. A. M. Gruenther, "Duplicate Bridge Complete," Bridge World Inc., New York, 1933.
6. J. D. Horton, R. C. Mullin, and R. G. Stanton, A recursive construction for Room designs, to appear.
7. R. C. Mullin and E. Nemeth, A counter-example to a multiplicative construction of Room squares, *J. Combinatorial Theory* **7** (1969), 264–265.
8. R. C. Mullin and E. Nemeth, An existence theorem for Room squares, *Canad. Math. Bull.* **12** (1969), 493–497.
9. R. C. Mullin and E. Nemeth, On furnishing Room squares, *J. Combinatorial Theory* **7** (1969), 266–272.
10. C. D. O'Shaughnessy, A Room design of order 14, *Canad. Math. Bull.* **11** (1968), 191–194.
11. T. G. Room, A new type of magic square, *Math. Gaz.* **39** (1955), 307.
12. K. R. Shah, Analysis of Room's square design, to appear.
13. R. G. Stanton and J. D. Horton, Composition of Room squares, *Proc. Coll. Combinatorial Math., Bolyai Janos Math. Soc., Budapest, 1969.*
14. R. G. Stanton and R. C. Mullin, Construction of Room squares, *Ann. Math. Statist.* **39** (1968), 1540–1548.
15. L. Weisner, A Room design of order 10, *Canad. Math. Bull.* **7** (1964), 377–378.